SOME EXAMPLES OF NONCOMMUTATIVE PROJECTIVE CALABI-YAU SCHEMES

YUKI MIZUNO

Abstract. In this article, we construct some examples of noncommutative projective Calabi-Yau schemes by using noncommutative Segre products and quantum weighted hypersurfaces. We also compare them with commutative Calabi-Yau varieties and examples in [14].

1. Introduction

Calabi-Yau varieties are very rich objects. They play an important role in mathematics and physics. In noncommutative geometry, (skew) Calabi-Yau algebras are often treated as noncommutative analogues of Calabi-Yau varieties. They have a deep relationship with quiver algebras ([10], [3]). Many known Calabi-Yau algebras are constructed by them. They are also used to characterize Artin-Schelter regular algebras ([25], [26]). In particular, a connected graded algebra A over a field k is Artin-Schelter regular if and only if A is skew Calabi-Yau.

On the other hand, a triangulated subcategory of the derived category of a cubic fourfold in $\mathbb{P}^5$ which is obtained by some orthogonal decompositions has the 2-shift functor [2] as the Serre functor. Moreover, the structure of Hochschild (co)homology is the same as that of a projective K3 surface ([15]). However, some of such categories are not obtained as the derived category of coherent sheaves of a projective K3 surface. They are called noncommutative K3 surfaces.

Artin and Zhang constructed a framework of noncommutative projective schemes which are defined from noncommutative graded algebras in [2]. In the framework, we can think of Artin-Schelter algebras as noncommutative analogues of projective spaces, which are called quantum projective spaces. Our objective is to produce examples of noncommutative projective Calabi-Yau schemes which are not obtained from commutative ones. As the definition of noncommutative projective Calabi-Yau schemes, we adopt the definition by Kanazawa ([14]). His definition is a direct generalization of the definition of (commutative) Calabi-Yau varieties to noncommutative projective schemes. He also constructed the first examples of noncommutative projective Calabi-Yau schemes as hypersurfaces of quantum projective spaces which are not isomorphic to commutative ones. Recently, some of the examples play an important role in noncommutative Donaldson-Thomas theory ([16], [17]).

In this paper, we construct new examples of noncommutative projective Calabi-Yau schemes by using noncommutative Segre products and weighted projective hypersurfaces. We detail that below. Many examples of Calabi-Yau varieties are known in algebraic geometry. Among them, it is well-known that some of them are complete intersections in Segre embeddings of products of projective spaces. Moreover, Reid gave a list of Calabi-Yau surfaces which are hypersurfaces in weighted projective spaces ([24], [13]). In this paper, we construct noncommutative analogues of the 2 types of examples of Calabi-Yau varieties. Although the methods by Kanazawa are efficient in our cases, we also need different approaches. In order to construct the former case, we perform a more detailed analysis of noncommutative projective schemes of $\mathbb{Z}^2$-graded algebras which are studied by Rompay ([33]). A different approach to noncommutative Segre products is also studied in [11]. For the latter case, we consider the quotients of weighted quantum polynomial rings. In commutative algebraic geometry, the projective spectrum $\mathrm{Proj}(k[x_0, \cdots, x_n])$ of a weighted polynomial ring is not necessarily isomorphic to $\mathrm{qgr}(k[x_0, \cdots, x_n])$, where $\mathrm{qgr}(k[x_0, \cdots, x_n])$ is the quotient category associated to $k[x_0, \cdots, x_n]$

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constructed in [2]. However, $\text{qgr}(k[x_0, \cdots, x_n])$ is thought of as a nonsingular model (see [28, Example 4.9]). We use this idea to construct other noncommutative projective Calabi-Yau schemes. In addition, it should be noted local structures of noncommutative projective schemes of quotients of weighted quantum polynomial rings are a little complicated. An analysis of the local structures was performed by Smith ([28]). Moreover, we show these are described by using the notion of quasi-Veronese algebras introduced by Mori ([20]). Those mentioned above are treated in Section 3.

We compare them with commutative projective Calabi-Yau varieties and the first examples in [14] in Section 4. We focus on noncommutative projective Calabi-Yau schemes of dimensions 2 and 3 in the section. We consider moduli spaces of point modules (closed points) of noncommutative schemes obtained in Section 3. However, weighted quantum polynomial rings are not generated in degree 1. So, the notion of point modules is not necessarily useful in this case. In this paper, we use the notion of closed points and theories concerning them in [29], [30] and [21], etc. Moreover, we also use methods in [22] and [23] to study simple modules of quantum affine spaces. A different approach to closed points of weighted quantum polynomial rings is studied in [31]. The notion of point modules in [31] corresponds to the notion of ordinary and thin points in [21]. We also show some of them are not isomorphic to any of commutative Calabi-Yau schemes and the first examples in [14]. To show they are not isomorphic, we also need Morita theory of noncommutative schemes in [5] (see also [2, Section 6]).

2. Preliminaries

Notation 2.1. In this article, k means an algebraically closed field of characteristic 0.

We suppose $\mathbb{N}$ contains 0.

Let A be a k-algebra, M be an A-bimodule and ψ, ϕ be algebra automorphisms of A. Then, we denote the associated A-bimodule by ${}^{\psi}M^{\phi}$, i.e. ${}^{\psi}M^{\phi} = M$ as k-modules and the new bimodule structure is given by $a * m * b := \psi(a)m\phi(b)$ for all $a, b \in A$ and all $m \in M$.

Definition 2.2 ([2, Section 2])**.** For any connected right Noetherian graded k-algebra $A = \bigoplus_{i=0}^{\infty} A_i$, we denote the category of graded right A-modules (resp. finitely generated graded right A-modules) by $\text{Gr}(A)$ (resp.$\text{gr}(A)$). We denote the shift functor by $(-)(1) : \text{Gr}(A) \to \text{Gr}(A), M \mapsto M(1) := \oplus M(1)_i := \oplus M_{i+1}$. When let $M, N \in \text{Gr}(A)$, $\text{Hom}_A(M, N) := \bigoplus_{n\in\mathbb{Z}} \text{Hom}_{\text{Gr(A)}}(M, N(n))$.

We also denote the subcategory of torsion modules in $\text{Gr}(A)$ (resp.$\text{gr}(A)$) by $\text{Tor}(A)$ (resp.$\text{tor}(A)$). We denote the quotient category $\text{Gr}(A)/\text{Tor}(A)$ (resp.$\text{gr}(A)/\text{tor}(A)$) by QGr(A) (resp.qgr(A)) and the canonical projection by $\pi : \text{Gr}(A) \to \text{QGr}(A)$. π has a right adjoint functor ω. Let $\mathcal{A} := \pi(A)$. The (general) projective scheme of A is defined as $\text{Proj}(A) := (\text{Qgr}A, \mathcal{A}, s)$. We also define the (Noetherian) projective scheme as $\text{proj}(A) := (\text{qgr}A, \mathcal{A}, s)$. Let $X := \text{proj}(A)$. The global section of any object $\mathcal{N}$ is $\text{H}^0(X, \mathcal{N}) = \text{Hom}_{\text{qgr}(A)}(\mathcal{A}, \mathcal{N})$. The cohomology is $\text{H}^i(X, \mathcal{N}) := \text{Ext}^i_{\text{qgr}(A)}(\mathcal{A}, \mathcal{N})$ for $i > 0$.

Definition 2.3 ([35, Section4], [32, Section 4])**.** Let A be a connected graded k-algebra and m_A be $A_{\geq 1}$. Let M be a right graded A-module.

Then, we denote $\lim_{n\to\infty} \text{Hom}_A(A/A_{\geq n}, M)$ by $\Gamma_{m_A}(M)$. By using this, we define a functor Γ_{m_A} : $\text{Gr}(A) \to \text{Gr}(A)$(we call Γ_{m_A} torsion functor). We denote the derived functor of Γ_{m_A} by $\text{R}\Gamma_{m_A}$ and $H^i\text{R}\Gamma_{m_A}$ by $H^i_{m_A}$. Note that if A is generated in degree 1, $\Gamma_{m_A} = \{m \in M \mid m_A^n m = 0 \text{ for some } n \in \mathbb{N}\}$.

Definition 2.4 ([35, Definition 3.3, 4.1], [32, Definition 6.1, 6.2])**.** Let A be a right and left Noetherian connected graded k-algebra and A°, A^e be the opposite algebra and the enveloping algebra of A, respectively. Let R be an object of $\text{D}^{\text{b}}(A^e)$. Then, R is called a dualizing complex of A if (1) R has a finite injective dimension over A and A°, (2) The cohomology of R is finitely generated as both A and A°-modules, (3) The natural morphism $A \to \text{RHom}_A(R, R)$ and $A \to \text{RHom}_{A^{\circ}}(R, R)$ are isomorphisms $\text{D}^{\text{b}}(A^e)$.

Moreover, R is called balanced if $\text{R}\Gamma_{m_A}(R) \simeq A'$ and $\text{R}\Gamma_{m_{A^{\circ}}}(R) \simeq A'$, where A' is the Matlis dual of A.

3. Calabi-Yau conditions

Definition 3.1 ([14, Section 2.2])**.** Let A be a connected right Noetherian graded k-algebra. Then, $\mathrm{proj}(A)$ is a projective Calabi-Yau n scheme if its global dimension is n and the Serre functor of the derived category $\mathrm{D^b}(\mathrm{qgr}(A))$ is the n-shift functor $[n]$.

3.1. **$\mathbb{Z}^2$-graded algebras and Segre products.** In commutative algebraic geometry, when let X be the Segre embedding of $\mathbb{P}^n \times \mathbb{P}^m$ into $\mathbb{P}^{nm+n+m}$, a smooth complete intersection $Y \subset X$ of bidegrees $(n+1, 0)$ and $(0, m+1)$ provides a Calabi-Yau variety. In this section, we construct noncommutative analogues of this example.

Let C be a connected $\mathbb{N}^2$-graded k-algebra. We denote the category of bigraded right C-modules (resp. finitely generated bigraded right C-modules) by $\mathrm{BiGr}(C)$ (resp.$\mathrm{bigr}(C)$). Let M be a $\mathbb{Z}^2$-graded right C-module. If $M_{(\geq s,\geq s)} := \bigoplus_{i\geq s, j\geq s} M_{ij} = 0$ for $s \gg 0$, then we say M is a torsion C-module. We denote the category of $\mathbb{Z}^2$-graded torsion C-modules by $\mathrm{Tor}(C)$. We also define $\mathrm{tor}(C)$ to be the intersection of $\mathrm{bigr}(C)$ and $\mathrm{Tor}(C)$. So, we can construct the quotient category $\mathrm{QBiGr}(C) := \mathrm{BiGr}(C)/\mathrm{Tor}(C)$ (resp.$\mathrm{qbigr}(C) := \mathrm{bigr}(C)/\mathrm{tor}(C)$) (cf. [33, Section 2]). We also denote the projection functor by π and its right adjoint functor by ω. We can define the projective scheme $\mathrm{proj}(C)$ associated to C and the notion of Calabi-Yau projective schemes as in the case of $\mathbb{N}$-graded algebra.

Moreover, we define $C_{++} := C_{(>0,>0)}$ and the torsion functor $\Gamma_{C_{++}}$ for a $\mathbb{N}^2$-graded k-algebra C to be the map which sends M to $\{m \in M | (C_{>n,>n})m = 0 \text{ for some } n \in \mathbb{N}\}$.

We also denote the maximal ideal of C by m_C and define the notion of dualizing complexes of C in the same way as in Section 2. Note that we define another torsion functor Γ_{m_c} to be the map which sends M to $\{m \in M \mid C_{>n}m = 0 \text{ for some } n \in \mathbb{N}\}$, where $C_{>n} := \{c \in C \mid \text{the total degree of } c > n\}$. See [25, Section 3] for details of Γ_{m_C}.

In this section, we prove the following theorem.

Theorem 3.2. *Let $A := k\langle x_0, \cdots, x_n\rangle/(x_jx_i - q_{ji}x_ix_j)_{i,j}$, $B := k\langle y_0, \cdots, y_m\rangle/(y_jy_i - q'_{ji}y_iy_j)_{i,j}$ and $C := A \otimes_k B$, where $q_{ij}, q'_{ij} \in k^\times$. Let $f := \sum_{i=0}^n x_i^{n+1}$ and $g := \sum_{i=0}^m y_i^{m+1}$ be two bihomogeneous polynomials of degree $(n+1, 0), (0, m+1)$, respectively. We assume that (i) $q_{ii} = 1, q_{ij}q_{ji} = 1$ and $q_{ij}^{n+1} = 1$, (ii) $q'_{ii} = 1, q'_{ij}q'_{ji} = 1$ and $q'^{m+1}_{ij} = 1$.*

Then, $\mathrm{proj}(C/(f,g))$ is a Calabi-Yau projective $(n+m-2)$ scheme if and only if $\prod_{i=0}^n q_{ij}$ and $\prod_{i=0}^m q'_{ij}$ are independent of j, respectively.

Note that f, g are central elements because of the choice of $\{q_{ij}\}, \{q'_{ij}\}$. To prove the theorem, we need to show some lemmas.

Lemma 3.3. *Let $\mathcal{R} := \pi(\mathrm{R}\Gamma_{C/(f,g)_{++}}(C/(f,g))^*)$ and $\mathcal{R}' := \pi(\mathrm{R}\Gamma_{m_{C/(f,g)}}(C/(f,g))^*)$. Then, the functors $- \otimes^{\mathbb{L}} \mathcal{R}$ and $- \otimes^{\mathbb{L}} \mathcal{R}'[-1]$ between $\mathrm{tail}(C/(f,g))$ and itself are natural isomorphic.*

Proof. First, let I_1, I_2 be the ideals generated by $m_{A/(f)}, m_{B/(g)}$ respectively. Then, we have $C/(f,g)_{++} = I_1 \cap I_2$ and $m_{C/(f,g)} = I_1 + I_2$ and have the following long exact sequence in $\mathrm{BiGr}(C^e)$

$$\cdots \to H^i_{m_{C/(f,g)}}(C/(f,g)) \to H^i_{I_1}(C/(f,g)) \oplus H^i_{I_2}(C/(f,g)) \to H^i_{C/(f,g)_{++}}(C/(f,g)) \to \cdots$$

by using the Mayer-Vietris sequence, where Γ_{I_k} is defined not by using degree of I_k but by using powers of I_k (i.e., $\Gamma_{I_k}(M) := \{m \in M \mid I_k^n m = 0 \text{ for some } n\}$). Note that we can use the Mayer-Vietris sequence in our case (see Remark 3.4 below). We also have the exact triangle

$$\mathrm{R}\Gamma_{m_{C/(f,g)}}(C/(f,g)) \to \mathrm{R}\Gamma_{I_1}(C/(f,g)) \oplus \mathrm{R}\Gamma_{I_2}(C/(f,g)) \to \mathrm{R}\Gamma_{C/(f,g)_{++}}(C/(f,g)).$$

This is obtained from the equivalences for $i = 0$ in Remark 3.4. Moreover, we have $H^i_{I_1}(C/(f,g))^*$ and $H^i_{I_2}(C/(f,g))^*$ are torsion modules for $C/(f,g)_{++}$ from Sub-Lemma 3.5. So, the cohomologies of $\mathrm{R}\Gamma_{I_1}(C/(f,g))^* \oplus \mathrm{R}\Gamma_{I_2}(C/(f,g))^*$ are torsion. Combining this with the above triangle, we obtain the claim.

∎

Remark 3.4. To use the Mayer-Vietris sequence, we need to prove the following equivalences

$$\lim_{n\to\infty} \mathrm{Ext}^i_{C/(f,g)}((C/(f,g))/({I_1}^n + {I_2}^n), -) \text{ and } H^i_{I_1+I_2}(-),$$
$$\lim_{n\to\infty} \mathrm{Ext}^i_{C/(f,g)}((C/(f,g))/I_1^n \oplus (C/(f,g))/I_2^n, -) \text{ and } H^i_{I_1}(-) \oplus H^i_{I_2}(-),$$
$$\lim_{n\to\infty} \mathrm{Ext}^i_{C/(f,g)}((C/(f,g))/(I_1^n \cap I_2^n), -) \text{ and } H^i_{I_1\cap I_2}(-).$$

In the commutative ring theory, these equivalences are proved by using cofinality and the Artin-Rees Lemma (cf. [4, Chapter 3] or [9, Chapter A1D]). In general, an ideal of a noncommutative ring does not satisfy the Artin-Rees Lemma. However, I_1, I_2 satisfy the Artin-Rees property in the sense of [18, Chapter 4.2] because I_1, I_2 are generated by normal elements in our case. Thanks to this fact, we prove the above equivalences in the same way as in the case of commutative rings.

Sub-Lemma 3.5. *Let I_1, I_2 be as in the proof of Lemma 3.3. $H^i_{I_1}(C/(f,g))^*$ and $H^i_{I_2}(C/(f,g))^*$ are torsion modules for $C/(f,g)_{++}$ for any i.*

Proof. We show that $H^i_{I_1}(C/(f,g))^*$ are torsion modules for $C/(f,g)_{++}$. First, we have $\Gamma_{I_1} = \Gamma_{I_1^{n+1}}$. Moreover, if let J_1 be the ideal generated by $x_0^{n+1}, \cdots, x_n^{n+1}$, then the we have $\Gamma_{I_1^{n+1}} = \Gamma_{J_1}$. So, it is enough to consider Γ_{J_1}. Note that $x_0^{n+1}, \cdots, x_n^{n+1}$ are central elements from the choice of $\{q_{ij}\}$. Moreover, we have a surjective localization map $N \to N[x_i^{-(n+1)}]$ for any x_i^{n+1} and any right injective $C/(f,g)$-module N. We also have $\Gamma_{J_1}(M)$ is an injective module for any $C/(f,g)$-module M because J_1 satisfies Artin-Rees property (see also the proof of Lemma 3.7). Thus, we can calculate the local cohomology for J_1 by using Čech complexes $\mathscr{C}(x_0^{n+1}, \cdots, x_n^{n+1}; C/(f,g))$ (cf. [9, Theorem A1.3], [19]). Then, we also have $\mathscr{C}(x_0^{n+1}, \cdots, x_n^{n+1}; C/(f,g)) = \mathscr{C}(x_0^{n+1}, \cdots, x_n^{n+1}; A/(f)) \otimes_k B/(g)$. This induces that $H^i_{I_1}(C/(f,g)) = H^i_{J_1}(C/(f,g)) = H^i_{m_A}(A/(f)) \otimes_k B/(g)$. Because $H^i_{m_A}(A/(f))_{>0} = 0$, $H^i_{I_1}(C/(f,g))^*$ is torsion for $C/(f,g)_{++}$.

□

Lemma 3.6. $\mathrm{gl.dim}(\mathrm{qgr}(C/(f,g))) = n + m - 2$.

Proof. We consider a bigraded (commutative) algebra $E := k[s_0, \cdots, s_n, t_0, \cdots, t_m]/(\sum_{i=0}^n s_i, \sum_{i=0}^m t_i))$ and the projective spectrum biProj(E) in the sense of [12, Section 1]. Then, an object in $\mathrm{qbigr}(C/(f,g))$ can be thought of as an object in the category of modules over the sheaf $\mathcal{A}$ of algebras, where $\mathcal{A}$ is the sheaf associated to algebras $(k[x_0, \cdots, x_n, y_0, \cdots, y_m]/(f,g)_{x_i y_j})_{(0,0)}$ for each open affine scheme $\mathrm{D}_+(s_i t_j) \simeq \mathrm{Spec}((E_{s_i t_j})_{(0,0)})$. Hence, it is enough to prove that the global dimension of $(k[x_0, \cdots, x_n, y_0, \cdots, y_m]/(f,g)_{x_i y_j})_{(0,0)} = n + m - 2$.

In our case, we can complete the rest of the proof in the same way as in [14, Section 2.3]. We give its sketch. For simplicity, we prove it when $i = j = 0$. Let $S_i := s_i/s_0, T_i := t_i/t_0, X_i := x_i/x_0, Y_i := y_i/y_0$. Then we consider the $k[S_1, \cdots, S_n, T_1, \cdots, T_m]/(1 + \sum_{i=1}^n S_i, 1 + \sum_{i=0}^m T_i)$-algebra $k\langle X_1, \cdots, X_n, Y_1, \cdots, Y_m\rangle/(X_iX_j - (q_{0i}q_{ij}q_{j0})X_jX_i, Y_iY_j - (q'_{0i}q'_{ij}q'_{j0})Y_jY_i, 1 + \sum_{i=1}^n X_i^{n+1}, 1 + \sum_{i=1}^m Y_i^{m+1})$, where the module structure is given by the identification $S_i = X_i^{n+1}, T_i = Y_i^{m+1}$. We denote the former by E and the latter by F. It is enough to prove that the global dimension of localization F_m of F at any maximal ideal $\tilde{m} := (S_1 - a_1, \cdots, S_n - a_n, T_1 - b_1, \cdots, T_m - b_m)$ of E with $1 + \sum_{i=1}^n a_i = 0, 1 + \sum_{i=0}^m b_i = 0$ is $n + m - 2$ ([14, Lemma 2.7]).

If any a_i, b_i is not 0, then $F/\tilde{m}$ is a twisted group ring and hence semisimple. Moreover, $S_1 - a_1, \cdots, S_n - a_n, T_1 - b_1, \cdots, T_m - b_m$ is a regular sequence in $F_{\tilde{m}}$. This induces the claim ([18, Theorem 7.3.7]). On the other hand, we assume that one of $\{a_1, \cdots, a_n, b_1, \cdots, b_m\}$ is 0. For example, we assume $a_1 = 0$. Then, we consider E/X_1. We repeat taking quotients and reduce to considering the global dimensions of the algebras $k[X, Y]/(X^{n+1} + 1, Y^{m+1} + 1)$, which are 0 ([18, Theorem 7.3.5]). □

Proof of Theorem 3.2. First, we calculate $\mathrm{R}\Gamma_{m_{C/(f,g)}}(C/(f,g))^*$. From [14, Proposition 2.4] (or [26, Example 5.5]) and the proof of [26, Lemma 6.1], we have

$$\mathrm{R}\Gamma_{m_C}(C))^* \simeq \mathrm{R}\Gamma_{m_A}(A)^* \otimes \mathrm{R}\Gamma_{m_B}(B)^* \simeq {}^{\phi}A^1(-n-1) \otimes {}^{\psi}B^1(-m-1)[n+m+2],$$

where ϕ(resp. ψ) is the graded automorphism of A(resp. B) which maps $x_j \mapsto \prod q_{ji}x_j$ (resp. $y_j \mapsto \prod q_{ji}y_j$). Then, we consider the distinguished triangle

$$\mathrm{R}\Gamma_{m_C}(C(-n-1,0)) \xrightarrow{\times f} \mathrm{R}\Gamma_{m_C}(C) \longrightarrow \mathrm{R}\Gamma_{m_{C/(f)}}(C/(f)),$$
$$\mathrm{R}\Gamma_{m_{C/(f)}}((C/(f))(0,-m-1)) \xrightarrow{\times g} \mathrm{R}\Gamma_{m_{C/(f)}}(C/(f)) \longrightarrow \mathrm{R}\Gamma_{m_{C/(f,g)}}(C/(f,g))$$

obtained from the exact sequences of C-bimodules

$$0 \longrightarrow C(-n-1,0) \xrightarrow{\times f} C \longrightarrow C/(f) \longrightarrow 0,$$
$$0 \longrightarrow (C/(f))(0,-m-1) \xrightarrow{\times g} C/(f) \longrightarrow C/(f,g) \longrightarrow 0.$$

Note that $\{f, g\}$ is a regular sequence in C. Hence, we have

$$\mathrm{R}\Gamma_{m_{C/(f,g)}}(C/(f,g))^* \simeq {}^{\phi}(A/(f))^1 \otimes_k {}^{\psi}(B/(g))^1[n+m].$$

In addition, we have the Serre duality in $\mathrm{D^b}(\mathrm{qbigr}(C/(f,g)))$ from Lemma 3.7 (we prove this below). Thus, $-\otimes^{\mathbb{L}} \pi(\mathrm{R}\Gamma_{C/(f,g)_{++}}(C/(f,g))^*)[-1]$ induces the Serre functor in $\mathrm{D^b}(\mathrm{qbigr}(C/(f,g)))$. Finally, from the above formula and Lemma 3.3, the Serre functor $-\otimes^{\mathbb{L}} \pi(\mathrm{R}\Gamma_{C/(f,g)_{++}}(C/(f,g))^*)[-1]$ induces the $[n+m-2]$-shift functor if and only if $\prod_{i=0}^{n} q_{ij}$ and $\prod_{i=0}^{m} q'_{ij}$ is independent of j (cf. [14, Remark 2.5]). This completes the proof.

□

Lemma 3.7 (Local Duality and Serre Duality)**.** *Let $D := C$ or $C/(f,g)$. We have the following.*

(1) Let $M \in \mathrm{D}^b(\mathrm{bigr}(D))$. Let $Q = \omega \circ \pi$. Then, we have

$$\mathrm{R}Q(M) \simeq \mathrm{RHom}_D(M, \mathrm{R}Q(D)')$$

in $\mathrm{D}^b(\mathrm{bigr}(D))$.

(2) Let $\mathcal{D} := \pi(D)$, $\mathcal{M} := \pi(M)$ and $\mathcal{R}_D := \pi(\mathrm{R}\Gamma_{D_{++}}(D)') \in \mathrm{D}^b(\mathrm{qbigr}(D))$. Then, we have

$$\mathrm{Ext}^i_{\mathrm{qbigr}(D)}(\mathcal{N}, \mathcal{M})^* \simeq \mathrm{Ext}^{-i}_{\mathrm{qbigr}(D)}(\mathcal{M}, (\mathcal{N} \otimes^{\mathbb{L}} \mathcal{R}_D)[-1]).$$

Moreover, the functor $(-\otimes^{\mathbb{L}} \mathcal{R}_D)[-1]$ induces an autoequivalence of $\mathrm{D}^b(\mathrm{qbigr}(D/(f,g)))$.

Proof. To prove (1), we want to apply [34, Theorem 0.4]. So, we show that the torsion class defined by Q is quasi-compact, finite dimensional and stable (about the definition, see [34, Definition 3.4]). We prove that the torsion class is stable. First, we prove the torsion class defined by D_{++} is stable. D_{++} is generated by normal elements $\{x_i y_j\}$. So, D_{++} has Artin-Rees property in the sense of [18, Chapter 4.2]. Thanks to this property, we apply the proof of [9, Lemma A1.4]. This shows the stability of the torsion class defined by D_{++}. On the other hand, we have the canonical exact sequence

$$0 \to \Gamma_{D_{++}}(I) \to I \to Q(I) \to \lim_{nto\infty} \mathrm{Ext}^1(D/D^n_{++}, I) \to 0.$$

When I is injective, we have $Q(I)$ is also injective from the stability of $\Gamma_{D_{++}}$. This shows the stability of the torsion class defined by Q.

Let $l := \mathrm{lcm}(n+1, m+1)$. Then, D_{++} and D^l_{++} define the same torsion class. Moreover, if let D' is the ideal generated by $\{x_i^l y_j^l\}_{i,j}$, then D' also gives the same torsion class. Thus, we can calculate the local cohomology for D' by using Čech complexes as in Sub-Lemma 3.5. This shows that the torsion class defined by D_{++} is quasi-compact and finite. For, the torsion class defined by Q, we can show the quasi-compactness and finiteness from the above exact sequence and the isomorphism

$$R^i Q(-) \simeq R^{i+1}\Gamma_{D_{++}}(-), \quad i \geq 1.$$

Hence, we can apply [34, Theorem 0.4]. Finally, we obtain the claim by taking dual.

About (2), we can prove the first statement in the same way as in [8, Theorem A.4] by using (1). The second statement follows from that $\mathrm{R}\Gamma_{D_{++}}(D)'$ is isomorphic to a shift of an invertible bimodule of D in $D(\mathrm{qbigr}(D))$. □

As a corollary of Theorem 3.2, we construct examples of noncommutative projective Calabi-Yau schemes by taking Segre products.

Definition 3.8. (1) The Segre product $A \circ B$ of A and B is the $\mathbb{N}$-graded k-algebra with $(A \circ B)_i = A_i \otimes_k B_i$.
(2) Let $M \in \mathrm{bigr}(C)$. We define a right graded $A \circ B$-module M_Δ as the graded $A \circ B$-module with $(M_\Delta)_i = M_{ii}$.

Lemma 3.9 ([33, Theorem 2.4])**.** *We have the following natural isomorphism*

$$\mathrm{qbigr}(C) \longrightarrow \mathrm{qgr}(A \circ B), \qquad \pi(M) \longmapsto \pi(M_\Delta).$$

In addition, the functor defined by $- \otimes_{A \circ B} C$ is the inverse of this equivalence.

Remark 3.10. We similarly obtain an equivalence

$$\mathrm{qbigr}(C/J) \simeq \mathrm{qgr}(A \circ B/J_\Delta),$$

where $J := (f, g) \in \mathrm{bigr}(C)$.

Combining Theorem 3.2 with Remark 3.10, we get the following.

Corollary 3.11. *Let $J := (f, g) \in \mathrm{bigr}(C)$. Then, $\mathrm{proj}(A \circ B/J_\Delta)$ is a projective Calabi-Yau scheme.*

3.2. **Weighted hypersurfaces.** Reid produced the list of all commutative weighted Calabi-Yau hypersurfaces of dimension 2 (for example, see [24], [13]). In this section, we construct noncommutative Calabi-Yau schemes from noncommutative weighted projective hypersurfaces. Let A be a connected graded k-algebra. Then the Veronese subring $B := A^{(r)}$ is the connected graded k-algebra $B = \bigoplus_i B_i := \bigoplus_i A_{rn}$. We consider the (commutative) weighted polynomial ring $A = k[x_0, \cdots, x_n]$ with $\deg(x_i) = a_i$. Then, $\mathrm{Coh}(\mathrm{proj}(A))$ is not equivalent to $\mathrm{qgr}(A)$, but to $\mathrm{qgr}(A^{(n+1)\mathrm{lcm}(a_0,\cdots,a_n)})$. However, we can think of $\mathrm{qgr}(A)$ as a resolution of $\mathrm{Coh}(\mathrm{Proj}(A))$ (cf. [28, Example 4.9]).

Theorem 3.12. *Let $w = (a_0, \cdots, a_n) \in \mathbb{Z}^{n+1}_{>0}$ and $d := \sum a_i$ such that d is divisible by a_i for any i. Let $C := k\langle x_0, \cdots, x_n\rangle/(x_j x_i - q_{ji} x_i x_j)_{i,j}$ be a quantum weighted polynomial ring, where $q_{ij} \in k^\times, \deg(x_i) = a_i$. Let $f := \sum x_i^{h_i}$, where $h_i := d/a_i$.*

We assume that $q_{ii} = 1, q_{ij} q_{ji} = 1$ and $q_{ij}^{h_i} = q_{ij}^{h_j} = 1$. Then, $\mathrm{proj}(C/(f))$ is a projective Calabi-Yau $(n-1)$ scheme if and only if there exists $c \in k$ such that $c^{a_j} = \prod_{i=0}^n q_{ij}$ for all j.

Remark 3.13. Theorem 3.12 is a generalization of [14, Theorem 1.1].

Lemma 3.14. *The balanced dualizing complex of $C/(f)$ is isomorphic to ${}^\phi(C/(f))^1[n]$, where ϕ is a graded automorphism of C which maps $x_i \mapsto \prod q_{ij} x_i$.*

Proof. Since C is Artin-Schelter regular, C is skew Calabi-Yau ([26, Lemma 1.2]). This induces that the balanced dualizing complex of C is isomorphic to ${}^\phi C^1(-d)[n+1]$, where ϕ is the Nakayama automorphism of C. So, we show that it is the map which maps $x_i \mapsto \prod q_{ij} x_i$. We use the idea of [26, Example 5.5]. When we think of C as a $\mathbb{Z}^{n+1}$-graded algebra, it is a $\mathbb{Z}^{n+1}$-graded twist (in the sense of [36]) of the commutative weighted polynomial ring $B = k[x_0, \cdots, x_n]$ with $(\deg(x_0), \cdots, \deg(x_n)) = ((a_0, 0, \cdots, 0), \cdots, (0, \cdots, 0, a_n))$ by $\sigma = (\sigma_0, \cdots, \sigma_n)$, where

$$\sigma_i(x_j) = \begin{cases} q_{ji}^{1/a_i} x_j & i < j \\ x_j & i \geq j \end{cases}.$$

Note that the twisted algebra does not depend on the choice of q_{ji}^{1/a_i}. Then, we use [26, Lemma 5.3, Theorem 5.4] and have

$$\phi = \sigma_0^{a_0} \circ \cdots \circ \sigma_n^{a_n} \circ \xi^{-1},$$

where $\xi : B \to B$ is the map which sends x_i to $q_{i+1i} q_{i+2i} \cdots q_{ni} x_i$. Hence, ϕ is the map above. The rest of the proof is done in the same way as in the proof of Theorem 3.2. $\square$

To prove the next lemma, we recall quasi-Veronese algebras. In detail, see [20, Section 3]. Let A be a $\mathbb{Z}$-graded ring. The k-th quasi-Veronese algebra $A^{[k]}$ of A is defined to be

$$A^{[k]} := \bigoplus_{i\in\mathbb{Z}} \begin{pmatrix} A_{ki} & A_{ki+1} & \cdots & A_{ki+k-1} \\ A_{ki-1} & A_{ki} & \cdots & A_{ki+k-2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{ki-k+1} & A_{ki-k+2} & \cdot & A_{ki}. \end{pmatrix}.$$

Then, we have $\mathrm{Gr}(A) \simeq \mathrm{Gr}(A^{[k]})$([20, Lemma 3.9]). The equivalence is obtained by the functor

$$Q : \mathrm{Gr}(A) \to \mathrm{Gr}(A^{[k]}), \quad M \mapsto \oplus_{i\in\mathbb{Z}}(\oplus_{j=0}^{k-1} M_{ki-j}).$$

So, Q induces an equivalence between $\mathrm{qgr}(A)$ and $\mathrm{qgr}(A^{[k]})$.

Lemma 3.15. $\mathrm{gl.dim}(\mathrm{qgr}(C/(f))) = n - 1$.

Proof. We use the idea of the proof in Lemma 3.6 below. We consider a graded k-algebra $D := k[s_0, \cdots, s_n]/(\sum_{i=0}^n s_i)$. Let $a = \max\{a_0, \cdots, a_n\}$. Note that $(C/(f))^{[a]}$ is generated in degree 1. So, from the above, we can think of $\mathrm{qgr}(C/(f))$ as the category of coherent modules over a sheaf $\mathcal{A}$ of $\mathcal{O}_D$-algebra, where $\mathcal{A}$ is the sheaf defined by a tiled matrix algebra

$$N_i = \begin{pmatrix} E_{i,0} & E_{i,1} & \cdots & E_{i,a-1} \\ E_{i,-1} & E_{i,0} & \cdots & E_{i,a-2} \\ \vdots & \vdots & \cdots & \vdots \\ E_{i,-a+1} & E_{i,-a+2} & \cdots & E_{i,0} \end{pmatrix}$$

on each $D_+(s_i)$, where $E_i := (C/(f))[x_i^{-1}]$ and $E_{i,l}$ is the degree l part of N_i. In addition, $R_1 := E_i \oplus E_i(1) \oplus \cdots \oplus E_i(a-1)$ and $R_2 := E_i \oplus E_i(1) \oplus \cdots \oplus E_i(a_i - 1)$ are progenerators in $\mathrm{Gr}(E_i)$. So, the categories of right $\mathrm{End}_{\mathrm{gr}}(R_1)$-modules and right $\mathrm{End}_{\mathrm{gr}}(R_2)$-modules are equivalent. We also have $\mathrm{End}_{\mathrm{gr}}(R_1) \simeq N_i$ and

$$\mathrm{End}_{\mathrm{gr}}(R_2) \simeq M_i := \begin{pmatrix} E_{i,0} & E_{i,1} & \cdots & E_{i,a_i-1} \\ E_{i,-1} & E_{i,0} & \cdots & E_{i,a_i-2} \\ \vdots & \vdots & \cdots & \vdots \\ E_{i,-a_i+1} & E_{i,-a_i+2} & \cdots & E_{i,0} \end{pmatrix}.$$

So, it is enough to prove the global dimensions of M_i is $n-1$ (cf. [28, Remark in page 2938]). For simplicity, we assume $i = 0$. We show the global dimension of $k[S_1, \cdots, S_n]/(1+\sum_{i=0}^n S_i)$-algebra M_0 is $n-1$. The module structure of M_0 is given by the identification $S_i = (x_i^{h_i}/x_0^{h_0})I_{a_0}$, where I_{a_0} is the $(a_0 \times a_0)$-identity matrix. Let $\tilde{m} = (S_1 - a_1, \cdots S_n - a_n)$ be a maximal ideal of D with $1+\sum_{i=1}^n a_i = 0$. We show $\mathrm{gl.dim}((M_0)_{\tilde{m}}) = n - 1$

If all a_i are not 0, then $M_0/\tilde{m}M_0$ is semisimple (note that this is equivalent to that the global dimension of $M_0/\tilde{m}M_0$ is 0). Actually, tha category of $M_0/\tilde{m}M_0$-modules is equivalent to the category of graded $E_0' := E_0/(x_1^{h_1}/x_0^{h_0} - a_1, \cdots, x_n^{h_n}/x_0^{h_0} - a_n)E_0$-modules. This is a Morita equivalence. We also have E_0' is a twisted group algebra obtained from a finitely generated abelian group of rank 1 when we forget its grading. In addition, A finitely generated abelian group is a product of a free abelian group and finite cyclic groups. These show that E_0' is isomorphic to a skew Laurent polyonimal ring $R[t, t^{-1}; \sigma]$, where R is a semisimple ring, $\sigma \in \mathrm{Aut}(R)$ and $\deg(t) > 0$. This induces the graded global dimension of E_0' is 0. So, we get $\mathrm{gl.dim}(M_0/\tilde{m}M_0) = 0$. Finally, $S_1 - a_1, \cdots, S_n - a_n$ is a regular seuence in $(M_0)_{\tilde{m}}$ and this induces the claim.

Suppose some of a_i are 0. For example, we assume $a_1 = 0$. Then, $(x_1^{h_1}/x_0^{h_0})I_{a_0}$ is an annihilator of any simple M_0-module N. We have a unique integer r_1 such that $0 \leq \deg(x_1/x_0^{r_1}) \leq a_0 - 1$. If

$\deg(x_1/x_0^{r_1}) = 0$ then, $J = x_1/x_0^{r_1} I_{a_0}$ annihilates N. Otherwise, the matrix

$$J = \begin{pmatrix} 0 & \cdots & \cdots & 0 & x_1/x_0^{r_1} & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & 0 & x_1/x_0^{r_1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & x_1/x_0^{r_1} \\ x_1/x_0^{r_1+1} & 0 & \cdots & 0 & \cdots & \cdots & \cdots & \vdots \\ 0 & x_1/x_0^{r_1+1} & \cdots & 0 & \cdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \cdots & \cdots & \cdots & \vdots \\ 0 & 0 & \cdots & x_1/x_0^{r_1+1} & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

annihilates N. So, it is enough to prove the global dimension of $M_0/JM_0 = n-2$. Repeating this process, we can reduce to considering the matrix algebra

$$M'_i = \begin{pmatrix} E'_0 & E'_1 & \cdots & E'_{a_0-1} \\ E'_{-1} & E'_0 & \cdots & E'_{a_0-2} \\ \vdots & \vdots & \cdots & \vdots \\ E'_{-a_0+1} & E'_{-a_0+2} & \cdots & E'_0 \end{pmatrix}$$

, where $E' := (k\langle x,y\rangle/(xy-qyx, x^{l_1}+y^{l_2}))[x^{-1}], (q \in k^*, l_1\deg(x) = l_2\deg(y))$. The global dimension of this algebra is 0 as above. Thus, we complete the proof. □

Proof of Theorem 3.12. The global dimension of $\mathrm{qgr}(C/(f))$ is finite. So, the balanced dualizing complex of $(C/(f))$ induces the Serre functor of $\mathrm{qgr}(C/(f))$ from [8, Theorem A.4]. Considering that the dualizing complex of $\mathrm{qgr}(C/(f))$ is ${}^{\phi}(C/(f))^1$, we complete the proof as in the proof of Theorem 3.2. □

4. Comparison and closed points

In this section, we calculate point modules (closed points) of noncommutative Calabi-Yau schemes obtained in Section 3 and compare our examples with commutative Calabi-Yau schemes and noncommutative Calabi-Yau projective schemes obtained in [14].

Definition 4.1 ([1, Definition 3.8], [27, Definition 3.1], [7])**.** Let A be a connected graded k-algebra generated in degree 1. Let M be a graded right A-module. We say M is a point module if M is cyclic, generated in degree 0 and $\dim_k(M_i) = 1$ for all $i \geq 0$.

We mention some basic facts about point modules of connected graded k-algebras which are needed in this section. For details, see [27, Section 3], [14, Section 3], etc.

Let $A := k\langle x_0, \cdots, x_n\rangle$ be a free associative algebra with $(n+1)$-variables. Then, the moduli space $\mathcal{M}_A$ of point modules of A is isomorphic to $\prod_{i=0}^{\infty} \mathbb{P}^n$. Let $M := \bigoplus_i km_i$ be a point module of A. If $m_i x_j = \alpha_{i,j} m_{i+1} (\alpha_{i,j} \in k^*)$, then we can describe the isomorphism between them as follows

$$\mathcal{M}_A \to \prod_{i=0}^{\infty} \mathbb{P}^n, \quad M \mapsto \{(\alpha_{i,0}, \cdots, \alpha_{i,n})\}_{i\in\mathbb{N}}.$$

Let $f := \sum a_x x \in A$ be a homogeneous element of degree d, where x means $x_{i_0} x_{i_1} \cdots x_{i_{d-1}}$. Then, the multilinearlization of f is an element $f^{\mathrm{mul}}$of the polynomial ring $k[y_{ij}]$ which is given by replacing x by $y_{0,i_0} y_{1,i_1} \cdots y_{d-1,i_{d-1}}$. Let $B := A/(f_1, \cdots, f_m)$, where f_i are homogeneous elements of degree d_i respectively . Then, the moduli spaces $\mathcal{M}_B$ of point modules of B is given by

$$\mathcal{M}_B = \{\{\alpha_i := (\alpha_{i,0}, \cdots, \alpha_{i,n})\}_{i\in\mathbb{N}} \in \mathcal{M}_A \mid f_k^{\mathrm{mul}}(\alpha_l, \cdots, \alpha_{l+d_k-1}) = 0, 1 \leq k \leq m, 0 \leq l\} \subset \mathcal{M}_A. \tag{4.0.1}$$

If $B = k\langle x_0, \cdots, x_n\rangle/(x_j x_i - q_{ji} x_i x_j)$ be a quantum polynomial ring with $(n+1)$-variables, then we have $\mathcal{M}_B \subset \mathbb{P}^n = \mathrm{Proj}(k[x_{0,0}, \cdots, x_{0,n}])$ and

$$\mathcal{M}_B \cap U_l = \bigcap_{i,j,k \neq l} \mathcal{Z}((q_{ij}q_{jk}q_{ki} - 1)t_i t_j t_k) \cap \bigcap_{i,j \neq l} \mathcal{Z}((q_{ij}q_{jl}q_{li} - 1)t_i t_j) \subset U_l \tag{4.0.2}$$

on the standard open affines $U_l = \mathrm{Spec}(k[t_0, \cdots, \hat{t_l}, \cdots, t_n])$ ($\hat{t_l}$ means omit t_l). We say $\{q_{ij}\}$ is special if $q_{ij}q_{jk}q_{ki} = 1$ for any three relation $\{x_i x_j - q_{ij} x_j x_i, x_j x_k - q_{jk} x_k x_j, x_k x_i - q_{ki} x_i x_k\}$. Otherwise, we say $\{q_{ij}\}$ is general (this definition is a little different from one in [14]).

4.1. **Segre products.** Here, we compute the moduli space $\mathcal{M}_{A \circ B/J_\Delta}$ of point modules of $A \circ B/J_\Delta$, where $A \circ B/J_\Delta$ is $\mathbb{N}$-graded connected k-algebra defined in Corollary 3.11.

In the above, point modules of $\mathbb{N}$-graded algebras are defined. Similarly, we can define point modules of $\mathbb{N}^2$-graded connected k-algebras. In particular, the moduli $\mathcal{M}_{A \otimes_k B}$ of point modules on $A \otimes_k B$ and the moduli $\mathcal{M}_{A \circ B}$ of point modules on $A \circ B$ are isomorphic to the fiber product $\mathcal{M}_A \times \mathcal{M}_B$ (see, Lemma 3.9 and [33, Corollary 2.10]). We also have an isomorphism between $\mathcal{M}_{A \otimes B/J}$ and $\mathcal{M}_{A \circ B/J_\Delta}$ from the commutativity of the following diagram

$$\begin{array}{ccc} \mathrm{qbigr}(A \otimes_k B) & \longrightarrow & \mathrm{qgr}(A \circ B) \\ \cup & \circlearrowright & \cup \\ \mathrm{qbigr}(A \otimes_k B/J) & \longrightarrow & \mathrm{qgr}(A \circ B/J_\Delta). \end{array}$$

In the following, we consider noncommutative Calabi-Yau projective 3 schemes obtained from $A \otimes B/J = k\langle x_0, x_1, x_2, x_3, y_0, y_1, y_2\rangle/(x_j x_i - q_{ji} x_i x_j, y_l y_k - q'_{lk} y_k y_l, f, g)_{i,j,k,l}$.

Proposition 4.2. *If $\{q_{ij}\}$ is general, then $\dim(\mathcal{M}_{A \otimes_k B/J}) = 1$. Therefore, $\dim(\mathcal{M}_{A \circ B/J_\Delta}) = 1$.*

Proof. If $\{q_{ij}\}$ is general, then $\mathcal{M}_A$ is the 1-skeleton $\mathcal{S}$ of $\mathbb{P}^3$ (cf. the proof of [14, Proposition 3.4]). Let U_i, V_i be standard open affines of $\mathrm{Proj}(k[x_{0,0}, x_{0,1}, x_{0,2}, x_{0,3}])$ and $\mathrm{Proj}(k[y_{0,0}, y_{0,1}])$, respectively. If $i = j = 0$, then

$$\mathcal{M}_{A \otimes_k B/J} \cap (U_i \times V_j) = ((U_i \cap \mathcal{S}) \times \mathbb{A}^1) \cap \mathcal{Z}(\tilde{f}^{\mathrm{mul}}),$$

where $\tilde{f}^{\mathrm{mul}}$ is the polynomial obtained from the multilinearlization of f, dehomogenezation and eliminating the variables except for $x_{0,i}, y_{0,i}$. This argument holds for any i, j. ∎

Proposition 4.3. *There exists a noncommutatative projective Calabi-Yau 3 scheme which is not isomorphic to any commmutative Calabi-Yau 3 variety.*

Proof. For example, we consider the noncommutatative Calabi-Yau 3 scheme in Theorem 3.2 defined by the quantum parameters

$$\mathbf{q} = (q_{ij}) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \quad \mathbf{q}' = (q'_{ij}) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \tag{4.1.1}$$

Then, $\mathrm{qbigr}(A \otimes_k B/J) \simeq \mathrm{qgr}(A \circ B/(J_\Delta))$ is equivalent to the category of coherent modules of a sheaf $\mathcal{A}$ of algebras on $X = \mathrm{biProj}(k[s_0, s_1, s_2, s_3, t_0, t_1, t_2]/(s_0 + s_1 + s_2 + s_3, t_0 + t_1 + t_2))$ (cf. the proof of Lemma 3.6). Moreover, we define a sheaf $\mathcal{Z}_\mathcal{A}$ to be the sheaf

$$\Gamma(U, \mathcal{Z}_\mathcal{A}) = \{s \in \Gamma(U, \mathcal{A}) \mid s|_V \in Z(\Gamma(V, \mathcal{A})), \forall V \subset U : \text{open}\}$$

for all U (cf.[5, Proposition 2.11]). In particular, if U is affine, then $\Gamma(U, \mathcal{Z}_\mathcal{A}) = Z(\Gamma(U, \mathcal{A}))$. Let $\phi : \mathrm{Spec}(\mathcal{Z}_\mathcal{A}) \to X$ be the natural morphism. Then, note that $\mathrm{qgr}(A \circ B/(f, g))$ is equivalent to the category of coherent modules of a sheaf $\phi^{-1}\mathcal{A}$ of algebra on $\mathrm{Spec}(\mathcal{Z}_\mathcal{A})$. Let W be a commutative Calabi-Yau scheme. If $\mathrm{qgr}(A \circ B/(f, g))$ is equivalent to $\mathrm{Coh}(W)$, then we have an isomorphism of schemes between $\mathrm{Spec}(\mathcal{Z}_\mathcal{A})$ and W by [5, Theorem 4.4] (see also [2, Section 6]). However, $\mathrm{Spec}(\mathcal{Z}_\mathcal{A}) \simeq X'_1 \times_k X'_2$, where $X'_1 \simeq \mathrm{Proj}(k[s'_0, s'_1, s'_2, s'_3]/({s'_0}^2 + {s'_1}^2 + {s'_2}^2 + {s'_3}^2)$ and $X'_2 \simeq \mathrm{Proj}(k[t'_0, t'_1, t'_2]/({t'_0}^3 + {t'_1}^3 + {t'_2}^3))$. This is because we have $Z(\Gamma(D_+(s_k), \tilde{A})) \simeq k[u_1, u_2, u_3]/(1 + u_1^2 + u_2^2 + u_3^2)$ and $Z(\Gamma(D_+(t_l), \tilde{B})) \simeq k[v_1, v_2]/(1 + v_1^3 + v_2^3)$ for any k and l. For example, note that when $k = 0$, we have $\Gamma(D_+(s_0), \tilde{A}) \simeq k\langle X_1, X_2, X_3\rangle/(X_iX_j - q''_{ij}X_jX_i, 1 + X_1^4 + X_2^4 + X_3^4)$, where $\tilde{A}$ is the sheaf associated to A on $\mathrm{Proj}(k[s_0, s_1, s_2, s_3]/(s_0 + s_1 + s_2 + s_3))$ and

$$\mathbf{q}'' = (q''_{ij}) = (q_{0i}q_{ij}q_{j0}) = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}.$$

This completes the proof. □

Remark 4.4. (1) In [14], the author constructs a noncommutative projective Calabi-Yau 3 scheme W whose moduli of point modules is finite points. We show that these noncommutative schemes are not isomorphic to the above one.

Proof : From Proposition 4.2, $A \circ B/(J_\Delta)$ has infinite point modules. We suppose that $W = \mathrm{proj}(k\langle w_0, w_1, w_2, w_3, w_4\rangle/(w_jw_i - q''_{ji}w_iw_j, w_0^5 + w_1^5 + w_2^5 + w_3^5 + w_4^5))$. If W and $\mathrm{proj}(A \circ B/(J_\Delta))$ are isomorphic, then each simple object of degree 1 is sent to a simple object of degree 1. The degree of a simple object $\mathcal{M}$ of a $\mathrm{proj}(A)$ is defined to be $\dim_k \mathrm{Hom}(\mathcal{O}_{\mathrm{proj}(A)}, \mathcal{M})$. Since $k\langle w_0, w_1, w_2, w_3, w_4\rangle/(w_jw_i - q''_{ji}w_iw_j, w_0^5 + w_1^5 + w_2^5 + w_3^5 + w_4^5)$ and $A \circ B/(J_\Delta)$ are generated in degree 1, any of their simple objects is given by a point module or a fat point module and the degrees of fat point modules are greater than 1 (cf. [6, Remark 6], [21, Section 3]). This induces a contradiction.

(2) When we consider $A \otimes B/(f)$ giving a projective Calabi-Yau 2 scheme, then the projective scheme is a twist of a commutative Calabi-Yau surface.

4.2. **Weighted hypersurfaces.** Let $C/(f)$ be the connected $\mathbb{N}$-graded k-algebra defined in Theorem 3.12. In this subsection, we focus on noncommutative Calabi-Yau projective surfaces. So, we suppose $n = 3$.

Example 4.5. Any weight (a_0, a_1, a_2, a_3) of noncommutative Calabi-Yau 2 schemes such that $\gcd(a_0, a_1, a_2, a_3) = 1$ is one of the following.

$$\begin{aligned}(a_0, a_1, a_2, a_3) = &(1,1,1,1), (1,1,1,3), (1,1,2,2), (1,1,2,4), (1,1,2,5), (1,1,4,6),\\ &(1,2,3,6), (1,3,3,4), (2,3,3,4), (1,2,6,9), (2,3,10,15), (1,6,14,21).\end{aligned}$$

4.2.1. *Closed points of weighted hypersurfaces.* In the rest of this subsection, we focus on the closed points of noncommutatative projective Calabi-Yau surfaces in Theorem 3.12 whose weights are of type $(1, 1, a, b)$ (cf. Example 4.5). We recall the notion of closed points of noncommutative projective schemes.

Definition 4.6 ([21, Definition 3.4], [29, Section 3.4])**.** Let A be a finitely generated connected right Noetherian graded k-algebra. A closed point of $\mathrm{proj}(A)$ is an object of $\mathrm{qgr}(A)$ represented by a 1-critical modules of A. In particular, if A is a graded quotient of (weighted) quantum polynomial rings, then every point is one of the following:

(1) An ordinary point, which is represented by a finitely generated 1-critical module of multiplicity 1.
(2) A fat point, which is represented by a finitely generated 1-critical module of multiplicity > 1.
(3) A thin point, which is represented by a finitely generated 1-critical module of multiplicity < 1.

Note that if A is generated in degree 1, the notion of ordinary points and the notion of point modules are the same and there is no thin point. We denote $\mid \mathrm{proj}(A) \mid$ by the set of closed points of proj(A).

We consider ordinary and thin points of $\mathrm{proj}(C) = \mathrm{proj}(k\langle x_0, x_1, x_2, x_3\rangle/(x_jx_i - q_{ji}x_ix_j))$ whose weight is of type $(a_0, a_1, a_2, a_3) = (1, 1, a, b) (a \le b)$. Since $a_0 = 1$, $C[x_0^{-1}]$ is strongly graded. So, from [21, Theorem 4.20], we have

$$\mid \mathrm{proj}(C) \mid = \mid \mathrm{spec}(C[x_0^{-1}]_0) \mid \bigsqcup \mid \mathrm{proj}(C/(x_0)) \mid,$$

where we denote $\mid \operatorname{spec}(C[x_0^{-1}]_0) \mid$ by the set of simple modules of $C[x_0^{-1}]_0$. In this equality, the 1-dimensional simple modules of $\operatorname{spec}(C[x_0^{-1}]_0)$ correspond to ordinary points and the $n(>1)$-dimensional simple modules of it correspond to fat points. Similarly, we have

$$\mid \operatorname{proj}(C) \mid = \mid \operatorname{spec}(C[x_0^{-1}]_0) \mid \bigsqcup \mid \operatorname{spec}(C/(x_0)[x_1^{-1}]_0) \mid \bigsqcup \mid \operatorname{proj}(C/(x_0,x_1)) \mid .$$

Since $C[x_0^{-1}]_0$ is a quantum polynomial ring generated by $x_i/x_0^{a_i}$, $C[x_0^{-1}]_0$ is isomorphic to a quantum polynomial ring $k\langle y_1,y_2,y_3\rangle/(y_jy_i - q'_{ji}y_iy_j)$, where $q'_{ji} = q_{0j}^{a_i}q_{ji}q_{i0}^{a_j}$. We also have $C/(x_0)[x_1^{-1}]_0$ is isomorphic to $k\langle z_2,z_3\rangle/(z_3z_2 - q_{13}^{a_2}q_{32}q_{21}^{a_3}z_2z_3)$. In addition, $C/(x_0,x_1) = k\langle x_2,x_3\rangle/(x_3x_2 - q_{32}x_2x_3)$ and it is known that a quantum weighted polynomial ring of 2 variables is some twisted algebra of the commutative weighted polynomial ring $k[x,y]$ with $\deg(x)=a, \deg(y)=b$ (for example, see [31, Example 4.1] or [36, Example 3.6]). So, it is enough to consider the closed points of $k[x,y]$ as for $\operatorname{proj}(C/(x_0,x_1))$.

Lemma 4.7. *Let $C_1 := C[x_0^{-1}]_0$ and $C_2 := C/(x_0)[x_1^{-1}]_0$. Then, we have a natural bijection between the set $\mid C_1 \mid_1$ of 1-dimensional simple modules of C_1 and $\mathbb{A}^3$ if $q'_{ji} = q_{0j}^{a_i}q_{ji}q_{i0}^{a_j} = 1$ for all i,j. We also have a natural bijection between the set $\mid C_2 \mid_1$ of 1-dimensional simple modules of C_2 and $\mathbb{A}^2$ if $q_{13}^{a_2}q_{32}q_{21}^{a_3} = 1$.*

Proof. The second half of the statements is well-known (for example, see [29, Section 4.3]). We show the first half of the statements.

From [22, Theorem A, B], we hava a surjective map from $\mathbb{A}^3$ to the set of all simple modules of C_1. Then, each element of $(k^*)^3 \subset \mathbb{A}^3$ is sent to a simple module of dimension $h := (\text{PI-deg}(C_1))^2$, where PI-deg$(C_1)$ means the PI degree of C_1. we can write q'_{ij} as ${q'}^{h_{ij}}$, where q' is a primitive d-th root and $h_{ij} \in \mathbb{Z}$. Then, h is the cardinality of the image of the homomorphism

$$\mathbb{Z}^3 \overset{(h_{ij})}{\to} \mathbb{Z}^3 \overset{p}{\to} (\mathbb{Z}/d\mathbb{Z})^3,$$

where p is the natural projection ([22, Proposition 2.3]). It is easy to see that $h=1$ if and only if $q'_{ij}=1$ for all i,j. So, if $q'_{ij}=1$, then any simple module of C_1 is isomorphic to a simple module of the form $C_1/(y_1-b_1, y_2-b_2, y_3-b_3)$ for some $b_1,b_2,b_3 \in k$. We also have $C_1/(y_1-b_1,y_2-b_2,y_3-b_3) \not\simeq C_1/(y_1-b'_1,y_2-b'_2,y_3-b'_3)$ if $(b_1,b_2,b_3) \neq (b'_1,b'_2,b'_3)$. Thus, we have the claim. ☐

We mention the closed points of $\operatorname{proj}(k[x,y])$ (we use proj in the sense of Definition 2.2) with $\deg(x)=a, \deg(y)=b$ such that $(a,b) = (2,2),(2,4)$ or $(4,6)$. Note that when $(a,b) = (1,1),(1,3)$ or $(2,5)$, the closed points of $\operatorname{proj}(k[x,y])$ is classified in [21, Theorem 3.16].

Lemma 4.8. *Let (a,b) and $R = k[x,y]$ be as above. Let $g := \gcd(a,b), a' := a/g$ and $b' := b/g$. Then, every closed point of $\operatorname{proj}(R)$ is one of the following:*

(1) $\pi R/(x)(-i),\ i = 0,\cdots,b-1$.
(2) $\pi R/(y)(-j),\ j = 0,\cdots,a-1$.
(3) $\pi R/(\beta x^{b'} - \alpha y^{a'})(-k)$, where $\alpha,\beta \in k^*$ and $k = 0,\cdots,g-1$.

Moreover, all of them are not isomorphic in $\operatorname{proj}(R)$. Any point of $\operatorname{proj}(R)$ is thin if $g>1$.

Proof. The proof is almost the same as the proof of [21, Lemma 3.15, Theorem 3.16]. We give the sketch of the proof. Firstly, every closed point of $\operatorname{proj}(R)$ is represented by a cyclic critical Cohen-Macaulay module of depth 1. Then, $M \in \operatorname{gr}(R)$ satisfies these and is generated in degree 0 if and only if M is isomorphic to one of $R/(x), R/(y)$ or $R/(\beta x^{b'} - \alpha y^{a'})$ $(\alpha,\beta \in k^*)$. Since being cyclic critical Cohen-Macaulay of depth 1 is invariant under shifting, any closed point is represented by some shifts of one of the above modules (that is, $R/(x)(-l), R/(y)(-l), R/(\beta x^{b'} - \alpha y^{a'})(-l),\ l \in \mathbb{Z}$). Finally, we classify the isomorphic classes of these modules in $\operatorname{proj}(R)$. In detail, we have no isomorphisms between the three types of closed points by using their Hilbert polynomial and multiplicity except in the case of $a=b$. Then, we can show $\pi R/(\beta x^{b'} - \alpha y^{a'}) \simeq \pi R/(\beta x^{b'} - \alpha y^{a'})(-gl),\ l \in \mathbb{Z}$ and $\pi R/(\beta x^{b'} - \alpha y^{a'}) \simeq \pi R/(\beta' x^{b'} - \alpha' y^{a'})$ if and only if $(\alpha,\beta) = (\alpha',\beta')$. In addition, we can show that $\pi R/(x) \simeq \pi R/(x)(-i)$ (resp. $\pi R/(y) \simeq \pi R/(y)(-j)$) if and only if $i \equiv 0 \pmod b$ (resp. $j \equiv 0 \pmod a$). From these, we get the claim. ☐

We study ordinary (and thin) points of weighted hypersurfaces by using the above investigation. In particular, we give examples of noncommutative projective Calabi-Yau schemes whose moduli of ordinary closed points are different from those in [14, Proposition 3.4] and commutative projective Calabi-Yau schemes.

Example 4.9. We consider the weight $(1,1,2,2)$ and the quantum parameter

$$\mathbf{q} = (q_{ij}) = \begin{pmatrix} 1 & 1 & 1 & \omega^2 \\ 1 & 1 & \omega^2 & 1 \\ 1 & \omega & 1 & 1 \\ \omega & 1 & 1 & 1 \end{pmatrix}, \quad \eta := \frac{-1+i\sqrt{3}}{2}.$$

Then, we have

$$\mathbf{q}' = (q'_{ij}) = \begin{pmatrix} 1 & \omega^2 & \omega \\ \omega & 1 & \omega^2 \\ \omega^2 & \omega & 1 \end{pmatrix}, \quad q_{13}^{a_2} q_{32} q_{21}^{a_3} = \omega^2.$$

This means that three out of x_0, x_1, x_2, x_3 are annihilators of M for any 1-critical module M of C. Since the ordinary or thin points are

$$\mid \mathrm{proj}(C) \mid_{\text{ord \& thin}} = \mid \mathrm{spec}(C[x_0^{-1}]_0) \mid_1 \bigsqcup \mid \mathrm{spec}(C/(x_0)[x_1^{-1}]_0) \mid_1 \bigsqcup \mid \mathrm{proj}(C/(x_0, x_1)) \mid,$$

the $\mid \mathrm{proj}(C/(f)) \mid_{\text{ord \& thin}}$ is 24 points. To be more precise, we have $\mid \mathrm{spec}((C/(f))[x_0^{-1}]_0) \mid_1 = \sqcup_{i \neq j} Z(y_i, y_j, 1 + y_1^6 + y_2^3 + y_3^3) \subset \mathbb{A}^3$, $\mid \mathrm{spec}(C/(f, x_0)[x_1^{-1}]_0) \mid_1 = \bigsqcup_{i=1,2} Z(z_i, 1 + z_2^3 + z_3^3)$ and $\mid \mathrm{proj}(C/(f, x_0, x_1)) \mid = \{3\text{pts}\} \sqcup \{3\text{pts}\} \subset \mathbb{A}^1 \sqcup \mathbb{A}^1$.

From Example 4.9 and a direct computation, we have the following proposition.

Proposition 4.10. *For a weight $(1,1,a,b)$ in Example 4.5 and a quantum parameter $\mathbf{q}$ which gives a projective Calabi-Yau scheme, if the set of ordinary and thin points of $\mathrm{proj}(C/(f))$ is finite, then the number of the set is always 24.*

The following proposition shows that some of noncommutative Calabi-Yau surfaces in Theorem 3.12 are new examples.

Proposition 4.11. *There exists a noncommutative Calabi-Yau projective surface which is obtained in Theorem 3.12 and not isomorphic to either commutative Calabi-Yau surfaces or noncommutative projective Calabi-Yau surfaces obtained in [14].*

Proof. For example, we choose the weight $(1,1,a,b)$ and the quantum parameter $\mathbf{q}$ as in Example 4.9. Then, the number of ordinary and thin points of $\mathrm{proj}(C/(f))$ is finite. So, we can show that $\mathrm{proj}(C/f)$ is not isomorphic to any commutative Calabi-Yau surfaces as in Remark 4.4.

We prove that $\mathrm{proj}(C/(f))$ is not isomorphic to any noncommutative Calabi-Yau surfaces in [14] by using the method of the proof of Proposition 4.3. First, note that we can think of $\mathrm{proj}(C/f)$ is the category of coherent modules of a sheaf $\mathcal{A}$ of algebras on $\mathrm{Proj}(k[s_0, s_1, s_2, s_3]/(s_0 + s_1 + s_2 + s_3))$ (cf. Lemma 3.15). Then, a direct computation shows $\mathrm{Spec}(\mathcal{Z}_{\mathcal{A}})$ has a singular point. In detail, $\mathrm{Spec}(Z(\Gamma(D_+(s_i), \mathcal{A})))$ is smooth when $i = 0, 1$ and not smooth when $i = 2, 3$.

We consider the weight $(1,1,1,1)$ and take a quantum parameter which gives a noncommutative Calabi-Yau surface whose moduli space of point modules is finite points. Then, the corresponding noncommutative scheme is also thought of as the category of coherent modules of a sheaf $\mathcal{B}$ of algebras on $\mathrm{Proj}(k[t_0, t_1, t_2, t_3]/(t_0 + t_1 + t_2 + t_3))$. In addition, $\mathrm{Spec}(\mathcal{Z}_{\mathcal{B}})$ is the projective plane or a smooth quadric hypersurface. This completes the proof. □

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Email address: m7d5932a72xxgxo@fuji.waseda.jp

Department of Mathematics, School of Science and Engineering, Waseda University, Ohkubo 3-4-1, Shinjuku, Tokyo 169-8555, Japan