Some examples of noncommutative projective Calabi-Yau schemes
(arXiv:2209.12190)

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A motivation

Notation k :alg closed fld of $\mathrm{ch}(k) = 0$.

Algebraic geometry $\cdots$ Study of **Varieties (Schemes)** $\approx$ " Zero loci of polynomials "

- Projective vars (schs) = closed subvars (subschs) of $\mathbb{P}^n$.
- Calabi-Yau mfds M = cpt sm vars with $\omega_M \simeq \mathcal{O}_M$.

An example of proj CY mfds

- $M \subset \mathbb{P}^4$ def by $x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 = 0$

Then, M is denoted by

$$\mathrm{Proj}(k[x_0, x_1, x_2, x_3, x_4]/(x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5)).$$

In general, we can define a proj sch Proj(R) for any comm graded algebra R.

Questions

1. Can we consider noncommutative (NC) proj schs for NC gr algs R ?
2. Can we also consider NC proj CY schs ?

This talk

We give the notion of NC proj CY schs and two types of examples.

- NC analogues of **hypersurs in weighted proj sps**
- NC analogues of **CI in products of proj sps**

Plan

- Introduction
- Definition of NC proj schemes
- NC proj CY schemes
- Result 1
- Comparison and examples
- Result 2

Notation

- $k = \overline{k}$: alg clo fld of $\mathrm{ch}(k) = 0$.

Introduction

- $R = \oplus_{i \geq 0} R_i$: a comm fin gen gr k-alg.
- $\mathrm{gr}(R)$: cat of fin gen gr R-mods.
- $\mathrm{fdim}(R)$: cat of fin dim gr R-mods.

Theorem (Serre)

Suppose that R is generated by R_1 as a k-algebra. Then,

$$\mathbf{Coh(Proj(}R\mathbf{))} \simeq \mathbf{qgr(}R\mathbf{)}\ (:= \mathbf{gr(}R\mathbf{)/\,fdim(}R\mathbf{)}).$$

Remark $\mathrm{qgr}(R)$ is the cat with

- $\mathrm{Obj}(\mathrm{gr}(R)) = \mathrm{Obj}(\mathrm{qgr}(R))$,
- $\mathrm{Hom}_{\mathrm{qgr}(R)}(\pi(M), \pi(N)) = \varinjlim_n \mathrm{Hom}_{\mathrm{gr}(R)}(M_{\geq n}, N_{\geq n})$

  ,where $\pi : \mathrm{gr}(R) \to \mathrm{qgr}(R)$ is the projection.

Remark

- When R is NOT generated by R_1, the thm does NOT necessarily hold.
- $\pi(M) \simeq \pi(N) \iff M_{\geq n} \simeq N_{\geq n}$ for $n \gg 0$.

Theorem (Gabriel, Rosenberg)

X, Y: noeth schemes
Then,

$$\mathbf{Coh}(\boldsymbol{X}) \simeq \mathbf{Coh}(\boldsymbol{Y}) \Rightarrow \boldsymbol{X} \simeq \boldsymbol{Y}.$$

Slogan

qgr(R) (or Coh(X)) is essential !

NC proj schemes

- $R = \oplus_{i \geq 0} R_i$: right noeth fin gen gr k-alg.
- $\text{qgr}(R) := \text{gr}(R)/\,\text{fdim}(R)$, which is the cat with

  obj same as the objs in $\text{gr}(R)$,

  mor $\text{Hom}_{\text{qgr}(R)}(\pi(M), \pi(N)) = \varinjlim_{n} \text{Hom}_{\text{gr}(R)}(M_{\geq n}, N_{\geq n})$.

Definition (NC proj schemes)

We call $\mathbf{(qgr(}\boldsymbol{R}\mathbf{), \boldsymbol{\pi}(}\boldsymbol{R}\mathbf{))}$ the projective scheme of R and denote it by $\mathbf{Proj_{nc}(}\boldsymbol{R}\mathbf{)}$.

Example

Let $q_{ij} \in k^\times$ for $0 \leq i, j \leq n$.

$$k[x_0, \cdots, x_n]_{(q_{ij})} := k\langle x_0, \cdots, x_n \rangle / (x_i x_j - q_{ij} x_j x_i)_{0 \leq i,j \leq n}.$$

We call this algebra a **quantum polyonomial ring**.

Remark

- $q_{ii} \neq 1 \Rightarrow x_i^2 = 0$.
- $q_{ij} q_{ji} \neq 1 \Rightarrow x_i x_j = x_j x_i = 0$.

NC proj CY schemes

Let X cpt sm var. X : CY $\overset{def}{\Longleftrightarrow} \omega_X \simeq \mathcal{O}_X$.

Definition

$\mathcal{D}$: k-lin tri cat. (e.g. $\mathrm{D}^b(X), \mathrm{D}^b(\mathrm{qgr(R)})$

A **Serre functor** of $\mathcal{D}$ is a funct $S_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}$ s.t.

- $S_{\mathcal{D}}$ is an equiv,
- $\mathrm{Hom}_{\mathcal{D}}(E, F) \simeq \mathrm{Hom}_{\mathcal{D}}(F, \mathcal{S}_{\mathcal{D}}(E))^{\vee}$.

※ A Serre funct is uniquely determined if it exists .

Fact

X : sm proj var of dim n.
Then, $- \otimes \omega_X[n] : \mathrm{D}^b(X) \to \mathrm{D}^b(X)$ is a Serre functor.

Remark X : CY $\Leftrightarrow \mathcal{S}_{\mathrm{D}^b(X)} \simeq [n]$.

Definition

$\mathcal{C}$: ab cat with enough injs. (e.g. $\mathrm{Coh}(X), \mathrm{qgr}(R)$)

$$\mathrm{gl.dim}(\mathcal{C}) := \mathrm{Sup}\{n \in \mathbb{Z} \mid \mathrm{Ext}^n_{\mathcal{C}}(E, F) \neq 0, {}^{\exists} E, F \in \mathrm{ob}(\mathcal{C})\}.$$

We call $\mathrm{gl.dim}(\mathcal{C})$ the **global dimension** of $\mathcal{C}$.

Fact

X : proj var.
Then, X is sm of dim $n \Leftrightarrow \mathrm{gl.dim}(\mathrm{Coh}(X)) = n$.

Definition

$\mathrm{Proj}_{\mathrm{nc}}(R) = (\mathrm{qgr}(R), \pi(R))$ is a **proj CY n-scheme** if

- $\mathrm{gl.dim}\ (\mathrm{qgr}(R)) = n$,
- $\mathcal{S}_{\mathrm{D}^b(\mathrm{qgr}(R))} \simeq [n]$.

Theorem (Kanazawa '14)

- $A := k[x_0, \cdots, x_n]_{(q_{ij})}/(x_0^{n+1} + \cdots + x_n^{n+1})$ *with* $\deg(x_i) = 1$.

Suppose

1. $q_{ii} = q_{ij}q_{ji} = 1, {}^{\forall} i, j$.
2. $q_{ij}^{n+1} = 1, {}^{\forall} i, j$.

Then,

$\mathrm{Proj}_{\mathrm{nc}}(A)$ *is a* **CY $(n-1)$-sch** *iff* $\prod_{i=0}^{n} q_{ij}$ *is independent of* j.

(i.e., ${}^{\exists} c \in k^{\times}$ *s.t.* $c = \prod_{i=0}^{n} q_{ij}$ *for* ${}^{\forall} j$*)*

Remark

- Thm of Kanazawa $\rightarrow$ NC analogue of Fermat hypersurs.
- $1, 2 \Rightarrow \mathrm{qgr}(A)$ is sm & $\mathcal{S}_{\mathrm{qgr}(A)}$ exists.
- $\prod_{i=0}^{n} q_{ij}$: indep of $j \Leftrightarrow \mathcal{S}_{\mathrm{qgr}(A)} \simeq [n-1]$.

Result 1

Theorem (M)

- $(d_0, \cdots, d_n) \in \mathbb{N}^{n+1}$ *satisfying* $d_i \mid d_0 + \cdots + d_n (=: d)$.
- $A := k[x_0, \cdots, x_n]_{(q_{ij})}/(x_0^{d/d_0} + \cdots + x_n^{d/d_n})$ *with* $\deg(x_i) = d_i$.

Suppose

1. $q_{ii} = q_{ij}q_{ji} = 1, \ ^{\forall} i, j$.
2. $q_{ij}^{d/d_i} = q_{ij}^{d/d_j} = 1, \ ^{\forall} i, j$.

Then,
$\mathrm{Proj_{nc}}(A)$ *is* **CY $(n-1)$-sch** *iff* $^{\exists} c \in k^{\times}$ *s.t.* $c^{d_j} = \prod_{i=0}^{n} q_{ij}$ *for* $^{\forall} j$.

Remark

- When $d_i = 1$, then the thm recovers Kanazawa's theorem.
- the thm is a NC analogue of weighted Fermat hypersurfaces.

Ideas of the proof

1. Proving qgr(A) is sm.
2. Calculating $\mathcal{S}_{\mathrm{qgr}(A)}$

About (1)

$C := k[y_0, \cdots, y_n]/(y_0 + \cdots + y_n) \hookrightarrow A \quad (y_i = x_i^{d/d_i})$.
Then,

$$\mathbf{qgr}(A) \simeq \mathbf{qgr}(A^{[d]})$$

↙ **This is called a quasi-Veronese alg by Mori**

$$\simeq \mathbf{Coh}(\widetilde{A^{[d]}}),\ A^{[d]} := \bigoplus_{i\in\mathbb{Z}} \begin{pmatrix} A_{di} & A_{di+1} & \cdots & A_{di+d-1} \\ A_{di-1} & A_{di} & \cdots & A_{di+d-2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{di-d+1} & A_{di-d+2} & \cdots & A_{di} \end{pmatrix}.$$

⇝ Enough to show gl.dim$((A^{[d]})_{(y_i)}) = n - 1$.
⇝ Taking a regular seq of $((A^{[d]})_{(y_i)})_{\mathfrak{n}} \quad ({}^{\forall}\mathfrak{n} \subset C_{(y_i)}$ maxi ideal$)$.

About (2)

↙ Yekutieli, Van den bergh

(a). $\mathcal{S}_{\mathrm{qgr}(A)} \simeq \pi(- \otimes^{\mathbb{L}}_A R\varGamma_{\mathfrak{m}}(A)')[-1]$.

- $\mathfrak{m} := A_{>0}$,
- $\varGamma_{\mathfrak{m}}(M) := \{m \in M \mid m\, A_{\geq n} = 0, {}^{\exists} n \in \mathbb{N}\}$: bimod
- $M' := \oplus_i \operatorname{Hom}_k(M_{-i}, k)$.

↙ Reyes, Rogalski and Zhang

(b). $R\varGamma_{\mathfrak{m}}(B)' \simeq {}^1B^{\mu}(-d)[n+1]$.

- $B := k[x_0 \cdots, x_n]_{(q_{ij})}$.
- $\mu : B \to B, x_j \mapsto \prod_{i=0}^{n} q_{ij} x_j$.
- mod struct of ${}^1B^{\mu}$ is def by $l * m * r := lm\mu(r)$.

(c). $\boldsymbol{R\varGamma_{\mathfrak{m}}(A)' \simeq {}^1A^{\mu}[n]}$.

(∵) Remember that $A = B/(f)$ where $f := \Sigma x_i^{d/d_i}$.

$0 \to B(-d) \xrightarrow{\times f} B \to A \to 0.$

$\rightsquigarrow R\varGamma_{\mathfrak{m}}(B)' \xrightarrow{\times f} R\varGamma_{\mathfrak{m}}(B)'(d) \to R\varGamma_{\mathfrak{m}}(A)'[1].$

Finally,

$$\mathcal{S}_{\mathrm{qgr}(A)} \simeq \pi(- \otimes_A {}^1 A^{\mu})[n-1]$$

So,

$$\mathcal{S}_{\mathrm{qgr}(A)} \simeq [n-1] \Leftrightarrow \pi(M^{\mu}) \simeq \pi(M) \quad ({}^{\forall} M \in \mathrm{gr(A)}).$$

$$\Leftrightarrow {}^{\exists} c \in k^{\times} \text{ s.t. } \prod_{i=0}^{n} q_{ij} = c^{d_j} \quad \text{for all } j \qquad (\star)$$

□

__Remark__

If $(\star)$ holds, $\varphi : M \to M$, $\varphi(m) = c^{\deg(m)} m$ is an iso with no dependence on M.

Definition

A **quasi-sch**/k is a pair $(\mathcal{C}, \mathcal{O})$, where

- k-lin ab cat $\mathcal{C}$. (e.g. $\mathrm{Coh}(X), \mathrm{qgr}(R)$)
- $\mathcal{O} \in \mathrm{Obj}(\mathcal{C})$. (e.g. $\mathcal{O}_X$, $\pi(R)$)

Definition

$(\mathcal{C}, \mathcal{O}), (\mathcal{C}', \mathcal{O}')$: quasi-schs/k.
Then,

- a **mor** from $(\mathcal{C}, \mathcal{O})$ to $(\mathcal{C}', \mathcal{O}')$ is a pair (F, φ) s.t. $F : \mathcal{C} \to \mathcal{C}'$ funct and $\varphi : F(\mathcal{O}) \overset{\simeq}{\to} \mathcal{O}'$.
- (F, φ) : **iso** $\Leftrightarrow$ F: equiv.

Example

$f : X \to Y$ mor (resp.iso) of schs/k.
Then, f induces a natural mor (resp.iso) of quasi-schs/k

$$f^* : (\mathrm{Coh}(Y), \mathcal{O}_Y) \to (\mathrm{Coh}(X), \mathcal{O}_X).$$

Comparison & examples

We set $n = 3$.
Then, (d_0, d_1, d_2, d_3) satifying $d_i \mid \sum d_i$ & $\gcd(d_i) = 1$ is one of

$$(1,1,1,1), (1,1,1,3), (1,1,2,2), (1,1,2,4), (1,1,4,6), (1,2,2,5),$$
$$(1,2,3,6), (1,2,3,9), (1,3,3,4), (1,6,14,21), (2,3,4,4), (2,3,10,15).$$

We choose $(1,1,2,2)$ and

$$(q_{ij}) := \begin{pmatrix} 1 & 1 & 1 & \omega^2 \\ 1 & 1 & \omega^2 & 1 \\ 1 & \omega & 1 & 1 \\ \omega & 1 & 1 & 1 \end{pmatrix}, \quad \omega := \frac{-1+\sqrt{3}i}{2}.$$

Proposition

Under the choice above,

- $\mathrm{Proj}_{nc}(A) \not\simeq (\mathrm{Coh}(M), \mathcal{O}_M)$, *($M$: comm CY).*
- $\mathrm{Proj}_{nc}(A) \not\simeq$ *" NC CY by Kanazawa ".*

(Sketch of the proof)
Let $(X, \mathcal{A})$: a pair of a noeth sch and a coh alg.
We define the sheaf of the center $\mathcal{Z}(\mathcal{A})$ of $\mathcal{A}$ as follows.

$$\mathcal{Z}(\mathcal{A})(U) := \{s \in \mathcal{A}(U) \mid s|_V \in Z(\mathcal{A}(V)),\ {}^\forall V \subset U \text{ open}\}.$$

Remark $\mathcal{Z}(\mathcal{A})(U) = Z(\mathcal{A}(U))$ if U is affine.

Proposition (Burdon, Brozd '22)

$(X, \mathcal{A}), (Y, \mathcal{B})$: pairs of noeth schs and coh algs.
Then, $\mathrm{Coh}(\mathcal{A}) \simeq \mathrm{Coh}(\mathcal{B}) \Rightarrow \mathrm{Spec}(\mathcal{Z}(\mathcal{A})) \simeq \mathrm{Spec}(\mathcal{Z}(\mathcal{B}))$.

- $\mathrm{Coh}(\mathcal{A}) \simeq \mathrm{Coh}(M) \Rightarrow \mathrm{Spec}(\mathcal{Z}(\mathcal{A})) \simeq M$.
- $(Y, \mathcal{B})$: NC CY 2-sch of Kanazawa $\Rightarrow \mathrm{Spec}(\mathcal{B})$: sm.
- However, $\mathrm{Spec}(\mathcal{Z}(\widetilde{A^{[2]}}))$ is not sm.

Remark
We can prove a part of the prop by comparing their point schemes.

Result 2

Fact

- $S := k[x_0 \cdots, x_n]$, $T := k[y_0 \cdots, y_m]$.

We regard $S \otimes_k T$ as a $\mathbb{Z}^2$-gr alg.
Let f_i be bihog polys in $S \otimes_k T$ $(i = 1 \cdots, r)$.
Then,

$$\textit{subsch def by } \{f_1, \cdots, f_r\} \simeq \mathrm{Proj}(\Delta(S \otimes T/(f_1, \cdots, f_r))).$$

※ $\deg(x_i) = (1, 0), \deg(y_i) = (0, 1)$.

※ For any $\mathbb{Z}^2$-gr alg R, $\Delta(R) := \bigoplus_{i \in \mathbb{Z}} R_{ii}$.

Theorem (M)

$X := \mathrm{Proj}_{\mathrm{nc}}(\Delta(S \otimes T/(f_1, f_2)))$.

(i)

- $S = k[x_0, \cdots, x_n]_{(q_{ij})}$.
- $T = k[y_0, \cdots, y_m]_{(q'_{ij})}$.
- $f_1 = \Sigma x_i^{n+1}, f_2 = \Sigma y_j^{m+1}$.

Suppose

1. $q_{ii} = q_{ij}q_{ji} = q_{ij}^{n+1} = 1$
2. $q'_{ii} = q'_{ij}q'_{ji} = {q'}_{ij}^{m+1} = 1$

Then,
X is **CY $(n+m-2)$-sch**
iff $^{\exists}c, c' \in k^{\times}$ *s.t.*
$c = \Pi_{i=0}^{n} q_{ij}, c' = \Pi_{i=0}^{m} q'_{ij}$.

(ii)

- $S = k[x_0, \cdots, x_n]_{(q_{ij})}$.
- $T = k[y_0, \cdots, y_{n+1}]$.
- $f_1 = \Sigma x_i^{n+1} y_i, f_2 = \Sigma y_j^{n+1}$.

Suppose

$$q_{ii} = q_{ij}q_{ji} = q_{ij}^{n+1} = 1$$

Then,
X is **CY $(2n-1)$-sch**
iff $^{\exists}c \in k^{\times}$ *s.t.* $c = \Pi_{i=0}^{n} q_{ij}$.

In (ii), a similar claim holds when $T = k[y_0, \cdots, y_n]$ and $f_2 = \Sigma y_i^n$.

Ideas of the proof

We use **qbigr(C) := bigr(C)/ fdim(C) ($C := S \otimes T/(f_1, f_2)$)**.

↙ Rompay

In our case, qbigr(C) $\simeq$ qgr($\Delta(C)$).

We show

1. qbigr(C) is sm. $\rightarrow$ We can prove as in MT1 (more easily).
2. Calculating $\mathcal{S}_{\mathrm{qbigr}(C)}$. $\rightarrow$ We prove **Key Lemma**.

Key Lemma

- $\mathfrak{m} := \oplus_{i,j>0} C_{i,j}$,
- $\Gamma_{\mathfrak{m}}(M) := \varinjlim \mathrm{Hom}(C/\mathfrak{m}^n, M)$.

(i) $\mathcal{S}_{\mathrm{qbigr}(C)} \simeq \pi(- \otimes^{\mathbb{L}} R\Gamma_{\mathfrak{m}}(C)')[-1]$.

(ii) $\pi(- \otimes^{\mathbb{L}} R\Gamma_{\mathfrak{m}}(C)') \simeq \pi(- \otimes^{\mathbb{L}} R\Gamma_{\mathfrak{m}'}(C)') \quad (\mathfrak{m}' := \oplus_{i+j>0} C_{i,j})$.

※ Calculating $R\Gamma_{\mathfrak{m}'}(C)'$ is easy than $R\Gamma_{\mathfrak{m}}(C)'$.

Thank you for listening !